A PLANE STEADY-STATE FREE-BOUNDARY PROBLEM FOR THE NAVIER–STOKES EQUATIONS

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The model problem of the plane slow steady-state motion of a viscous incompressible fluid with a free boundary is investigated. It is assumed that the free boundary does not have any points in common with the solid surfaces confining the fluid. By the solution of the auxiliary fixed-boundary problem for the Navier–Stokes equations the problem is reduced to an operator equation describing the form of the free surface. The existence and uniqueness problems for the solution and its qualitative behavior are analyzed.

1. Statement of the Problem

Consider the plane steady-state motion of a viscous incompressible fluid in a curvilinear two-dimensional channel, the upper boundary of which is free, while the lower boundary (bottom) represents a solid rectilinear wall on which there are periodically distributed regions of fluid ingress and egress. We introduce dimensionless variables, referring distances to the average depth h of the fluid, velocities to $h^{-1}\nu$ (ν is the kinematic viscosity coefficient), and the pressure to $\rho h^{-2}\nu^2$ (ρ is the density of the fluid). Then, the equations of motion are written in the form (mass forces are absent)

$$\Delta \mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0 \tag{1.1}$$

in the strip $-\infty < x_1 < \infty$, $-1 < x_2 < f(x_1)$. Here, $x_2 = f(x_1)$ is the equation for the free surface, which by assumption is projected one-to-one onto the bottom $x_2 = -1$. We seek solutions that are periodic in

$$\mathbf{v}(x_1 + l, x_2) \equiv \mathbf{v}(x_1, x_2), \; p(x_1 + l, x_2) \equiv p(x_1, x_2), \; f(x_1 + l) \equiv f(x_1) \tag{1.2}$$

We denote by Ω the "rectangle" $0 < x_1 < l$, $-1 < x_2 < f$, with the bottom leg $\Sigma = \{x_1, x_2 : 0 < x_1 < l, x_2 = -1\}$, and the curvilinear upper leg $\Gamma = \{x_1, x_2 : 0 < x_1 < l, x_2 = f(x_1)\}$; $\bar{\Omega}$ denotes the closure of Ω.

We denote by $\mathbf{n}$ the outward-normal unit vector, and by $\boldsymbol{\tau}$ the unit vector tangent to the free surface. The boundary conditions at the free surface have the form

$$\mathbf{v}|_\Gamma\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot T|_\Gamma\cdot\boldsymbol{\tau} = 0 \tag{1.3}$$

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right) = \mu\mathbf{n}\cdot T|_\Gamma\cdot\mathbf{n} \tag{1.4}$$

$$T_{ij} = -p\delta_{ij} + 2S_{ij}, \quad i, j = 1,2, \quad 2S_{ij} = \partial v_i / \partial x_j + \partial v_j / \partial x_i$$

Here, T is the stress tensor, S_{ij} is an element of the strain rate tensor, the prime signifies differentiation with respect to x_1, and $\mu = \rho\nu^2(\sigma h)^{-1}$ is a dimensionless parameter related to the coefficient of surface tension σ. The first condition (1.3) signifies the absence of fluid flow across the free boundary, and the second condition states that the tangential component of the stress vector acting on the free surface

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must vanish. Condition (1.4) states that the normal component of the stress vector is equal to the surface pressure, which according to the Laplace equation is proportional to the curvature of the free surface.

Requiring that the dimensionless average depth of the fluid be equal to unity, we find that

$$\int_0^l f dx_1 = 0 \tag{1.5}$$

We also require fulfillment of the inequality $|f| \le \delta < 1$ for all x_1 (δ = const > 0), which precludes the possibility of contact of the free surface with the bottom.

Finally, for the system (1.1), we impose the following boundary condition on the bottom:

$$\mathrm{v}|_\Sigma = \mathrm{w}(x_1) \tag{1.6}$$

Here, w is a given vector function, l-periodic in x, such that the restriction of w to $[0, l]$ is a function in the Hölder class $\mathbf{C}^{2+\alpha}$, $0 < \alpha < 1$, finite in $(0, l)$. (See, e.g., [1] for the definition of the Hölder spaces $C^{m+\alpha}$, integer-valued $m \ge 0$.) If $\varphi(x) \in C^{m+\alpha}(\overline{Q})$, where $\overline{Q}$ is a closed bounded domain, then, $|\varphi|_{m+\alpha\overline{Q}}$ denotes the norm of φ in $C^{m+\alpha}$. The expression $\mathbf{w} \in \mathbf{C}^{2+\alpha}[0, l]$ signifies that every component of the vector $\mathbf{w}$ is a member of $C^{2+1}[0, l]$. The steady-state requirement yields the additional condition

$$\int_0^l w_2 dx_1 = 0 \tag{1.7}$$

which connotes zero total fluid flow across each "cell" of the bottom.

The stated problem calls for the determination of twice continuously differentiable functions $\mathbf{v}(x_1, x_2)$ and $f(x_1)$, as well as a continuously differentiable function $p(x_1, x_2)$, all of which satisfy relations (1.1)-(1.6).

We note that the division of the conditions at the free boundary into two groups (1.3) and (1.4) is not merely fortuitous. It is dictated by the proposed method of solving problem (1.1)-(1.6). In the first stage the form of the free surface is fixed. For a fixed f, the boundary-value problem with conditions (1.2), (1.3), and (1.6) is solved for the system (1.1). We refer to this problem hereinafter simply as the auxiliary problem. A remarkable feature of the auxiliary problem is the fact that it is solvable "in the large," i.e., without any constraints on the initial data.

From the solution of the auxiliary problem, we determine $\mathbf{n} \cdot \mathrm{T}|_\Gamma \cdot \mathbf{n}$ and substitute the result into the other condition (1.4) at the free boundary. The relation so obtained can be treated as an equation for the determination of a curve of specified curvature. Inverting the curvature operator with conditions (1.2) and (1.5), we arrive at an equation $f = F(f)$, where F is a nonlinear continuous operator in a certain Banach space. We prove that for small $|\mathrm{w}|_{2+\alpha}$ this equation has a solution unique "in the small."

An essential feature of the proposed approach to the stated problem with an unknown boundary is the introduction of the surface tension into the boundary condition on the free surface. It is, in fact, this feature that enables us to reduce problem (1.1)-(1.6) to a manageable operator equation. It is important to mention that the role of the surface tension as a regulating factor in free-boundary problems has been indicated earlier by Garipov [2] and Shcherbina [3] (as communicated to the author by V. Kh. Izakson, the introduction of surface tension in the problem of the initiation of thermal convection in a fluid layer with a free boundary makes it possible to reduce it to the classical problem of finding the bifurcation points of a completely continuous operator).

2. Fundamental Definitions and Inequalities

Consider the space $L_2(\Omega)$ formed by two-dimensional vector functions l-periodic in x_1 whose components are square summable over a domain Ω. The scalar product in $L_2(\Omega)$ is defined by the equation

$$(\mathrm{u}, \mathrm{v}) = \int_\Omega \mathbf{u} \cdot \mathrm{v} dx$$

and the norm $||\mathbf{u}|| = (\mathbf{u}, \mathbf{u})^{1/2}$.

We bring in the operator $\mathcal{A}$, which establishes a correspondence between the solutions $\mathbf{u}(x)$ of the linear problems

$$-\Delta\mathbf{u} + \nabla p = \boldsymbol{\zeta}(x), \quad \nabla\cdot\mathbf{u} = 0, \quad x \in \Omega \tag{2.1}$$

$$\mathbf{u}(x_1 + l, x_2) \equiv \mathbf{u}(x_1, x_2), \quad \mathbf{u}|_\Sigma = 0, \quad \mathbf{u}|_\Gamma\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot T|_\Gamma\cdot\boldsymbol{\tau} = 0 \tag{2.2}$$

and their free terms $\zeta(x)$, namely $\mathcal{A}\mathbf{u} = \zeta$. The domain of definition D_A of the operator $\mathcal{A}$ consists of solenoidal vector functions $\mathbf{u} \in C^2(\Omega)$ $C^1(\overline{\Omega})$ subject to the boundary conditions (2.2). The operator $\mathcal{A}$ is symmetric, since for $\mathbf{u}, \mathbf{v} \in D_{\mathcal{A}}$

$$(\mathcal{A}\mathbf{u}, \mathbf{v}) = \int_\Omega (-\Delta\mathbf{u} + \nabla p)\cdot\mathbf{v}\,dx = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)dx \equiv 2\int_\Omega \sum_{i,j=1}^{2} S_{ij}(\mathbf{u})\,S_{ij}(\mathbf{v})\,dx$$

In the derivation of the latter equation the Green's formula for the Stokes system [1] is used. The surface integrals resulting therefrom vanish due to conditions (2.2). On the set, we introduce the new scalar product, asserting by definition that

$$[\mathbf{u}, \mathbf{v}] = \frac{1}{2}(\mathcal{A}\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^{2} S_{ij}(\mathbf{u})\,S_{ij}(\mathbf{v})\,dx \tag{2.3}$$

It is easily shown that all axioms of the scalar product are satisfied for (2.3). In particular, if $[\mathbf{u},\mathbf{u}] = 0$, then, $S_{ij}(\mathbf{u}) = 0$ for $x \in \overline{\Omega}$. We infer from the latter considerations and from $\mathbf{u}|_\Sigma = 0$ that $\mathbf{u} = 0$ in $\overline{\Omega}$.

Completing the set $D_{\mathcal{A}}$ on the norm $\|\mathbf{u}\|$, we obtain a subspace of $\mathbf{L}_2(\Omega)$, which we denote by $\mathbf{J}(\Omega)$. We denote by $\mathbf{G}(\Omega)$ the orthogonal complement to $J(\Omega)$ in $\mathbf{L}_2(\Omega)$. Following [1], we can show that $G(\Omega)$ consists of $\nabla\varphi$, where $\varphi \in W_2^1(\Omega)$ (see, e.g., [4] for the definition and properties of the Sobolev spaces W_r^m). If $\varphi(x) \in W_r^m(\Omega)$, then, the symbol $\|\varphi\|_r^{(m)}$ denotes the norm of φ in W_r^m. If $m = 0$, the superscript in the notation $\|\varphi\|_r^{(0)}$ is dropped.

The completion of the set $D_{\mathcal{A}}$ on the norm $|||\mathbf{u}||| = [\mathbf{u}, \mathbf{u}]^{1/2}$ yields a Hilbert space, which we call the energy space of the operator $\mathcal{A}$ and denote by $\mathbf{H}(\Omega)$. Note that the elements of $H(\Omega)$ satisfy "on the average" all the boundary conditions (2.2) except the last: $\mathbf{n}\cdot T|_\Gamma\cdot\tau = 0$. In the terminology of [5], the latter condition is natural for the differential operator $\mathcal{A}$, and the remaining conditions (2.2) are principal conditions.

By virtue of (2.3), the space $\mathbf{H}(\Omega)$ is a subspace of the vector space $W_2^1(\Omega)$ with norm

$$\|\mathbf{u}\|_2^{(1)} = \left[\int_\Omega (|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2)\,dx\right]^{1/2}$$

An exceedingly important consideration is the fact that the norms $\|\mathbf{u}\|_2^{(1)}$ and $|||\mathbf{u}|||$ are equivalent. It is clear that $|||\mathbf{u}||| \le C\|\mathbf{u}\|_2^{(1)}$, where C can be evaluated as 2. The proof of the converse inequality between the norms in H and W_2^1 is based on two inequalities that hold for $\mathbf{u}$ in $\mathbf{H}(\Omega)$. The first,

$$\int_\Omega |\mathbf{u}|^2\,dx \leqslant C_1 \int_\Omega |\nabla\mathbf{u}|^2\,dx \tag{2.4}$$

is proved as in [1]; C_1 can be evaluated as the number $(1+\delta)^2$, where $\delta = \max|f|$ (here, and elsewhere, the quantities C_k, $k = 1, 2, 3, \ldots$, denote positive constants). In the proof of (2.4), use is made of the density of $D_{\mathcal{A}}$ in $\mathbf{H}(\Omega)$, and the fact that $\mathbf{u}|_\Gamma = 0$ for $\mathbf{u} \in D_{\mathcal{A}}$.

The second inequality is a variant of the well-known Korn inequality in the theory of elasticity; for any $\mathbf{u}$

$$\int_\Omega |\nabla\mathbf{u}|^2\,dx \leqslant C_2 \int_\Omega \sum_{i,j=1}^{2} S_{ij}^2(\mathbf{u})\,dx \equiv C_2|||\mathbf{u}|||^2 \tag{2.5}$$

where C_2 depends only on the domain Ω. Inequalities of the type (2.5) have been verified for various subspaces of W_2^1 by A. Korn, as well as by Friedrichs [6], Éidus [7, 8], and others. Due to the space limitations of the article, we do not give the proof of (2.5). We merely point out that it is very much like the proof in [7]. On the basis of (2.4) and (2.5), we obtain

$$\|u\|_2^{(1)} \leqslant C_3 |u|, \quad \|u\| \leqslant C_4 |||u|||, \quad (C_3 = [(1 + C_1) C_2]^{1/2}, \quad C_4 = (C_1 C_2)^{1/2}).$$

The latter inequality implies that the operator $\mathcal{A}$ is positive definite [5]. It follows from inequality (2.5) and a theorem of Rellich [1] that any bounded set in $\mathbf{H}(\Omega)$ is compact in $\mathbf{J}(\Omega)$.

The foregoing properties of the operator $\mathcal{A}$ and its energy space $\mathbf{H}(\Omega)$ enable us to prove the existence of the eigenfunctions of the operator. The eigenfunctions $\mathbf{e}_k$, $k = 1, 2, 3, \ldots$, are the solutions of the problems

$$-\Delta \mathbf{e}_k + \nabla q_k = \lambda_k \mathbf{e}_k, \quad \nabla \cdot \mathbf{e}_k = 0, \quad x \in \Omega$$
$$\mathbf{e}_k (x_1 + l, x_2) \equiv \mathbf{e}_k (x_1, x_2)$$
$$\mathbf{e}_k|_\Sigma = 0, \quad \mathbf{e}_k|_\Gamma \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot T(\mathbf{e}_k)|_\Gamma \cdot \tau = 0$$

Invoking a variational method in accordance with the scheme described in [5], we deduce the following results. The operator $\mathcal{A}$ has an infinite set of eigenvalues $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n \leq \cdots, \lambda_n \to \infty$ as $n \to \infty$. The inverse of $\mathcal{A}$ is continuous. The eigenfunctions of $\mathcal{A}$ form a system that is complete and orthogonal both in $\mathbf{J}(\Omega)$ and in $\mathbf{H}(\Omega)$. The functions $\mathbf{e}_k$ are infinitely many times differentiable in Ω. Their smoothness in a closed domain is determined by the smoothness of Γ. If $f \in C^{m+1+\alpha}[0, l]$, $m \geq 1$, the functions $\mathbf{e}_k$ are of the Hölder class $\mathbf{C}^{m+\alpha}(\bar{\Omega})$. These functions as solutions of certain variational problems satisfy the natural boundary condition for the operator $\mathcal{A}$, $\mathbf{n} \cdot T|_\Gamma \cdot \tau = 0$ in the customary sense.

Next, we consider inequalities (2.4) and (2.5). According to [1, 6–8], they hold for domains having a piecewise-smooth boundary. We examine the family of domains Ω bounded above by various curves $\Gamma: x_2 = f(x_1)$, $0 \leq x_1 \leq l$. We assume that the functions f are bounded in the aggregate on the norm of $C^{1+\alpha}[0, l]$, so that $|f|_{1+\alpha} \leq \delta < 1$. It then turns out that the constants C_1 and C_2 in inequalities (2.4) and (2.5) can be chosen so as to be independent of the domain Ω (we also assume that this choice has already been made). The foregoing assertion is proved by a simple contradiction argument.

3. Generalized Solution of the Auxiliary Problem

In this section, we prove the existence of a generalized solution of problem (1.1)–(1.3), (1.6). We assume that the curve Γ is specified by functions $f(x_1) \in C^{1+\alpha}[0, l]$ and $|f|_{1+\alpha} \leq \delta < 1$.

LEMMA 3.1. Let there be specified on $[0, l]$ a function $\mathbf{w}(x_1)$, finite in $(0, l)$, and satisfying (1.7), such that $\mathbf{w} \in \mathbf{C}^{m+\alpha}$, integer-valued $m \geq 2$. Then, a vector function $\mathbf{a}(x)$ exists, such that

$$\mathbf{a}(x_1, -1) = \mathbf{w}(x_1), \quad \mathbf{a} \in \mathbf{C}^{m+\alpha}(\bar{\Omega}), \quad \nabla \cdot \mathbf{a} = 0 \quad \text{for} \quad x \in \Omega$$

$\mathbf{a}$ vanishes outside the rectangle $-1 \leq x_2 \leq -(1 + \delta)/2$, $x_1 \in \operatorname{supp} \mathbf{w}$, and for any $\mathbf{u} \in \mathbf{H}(\Omega)$

$$\left| \int_\Omega \mathbf{a} \cdot \mathbf{u} \cdot \nabla \mathbf{u} dx \right| \leqslant \frac{1}{C_2} \int_\Omega |\nabla \mathbf{u}|^2 dx \tag{3.1}$$

Here, C_2 is the constant in inequality (2.5), and supp w denotes the support of w (see [9, 10, 1] for the proof of the lemma). The indicated proofs yield the explicit construction of a solenoidal continuation $\mathbf{a}$ of the vector w into the domain Ω. The situation can also be so arranged such that

$$|\mathbf{a}|_{2+\alpha,\bar{\Omega}} \leqslant C_5 |\mathbf{w}|_{2+\alpha,[0,l]}.$$

We choose and fix for all time one of the continuations $\mathbf{a}(x)$ according to Lemma 3.1.

A generalized solution of the auxiliary problem is a function $\mathbf{v}(x)$ such that $\mathbf{v} - \mathbf{a} = \mathbf{u} \in \mathbf{H}(\Omega)$, and for any $\mathbf{\Phi} \in \mathbf{H}(\Omega)$, the following identity holds:

$$2[\mathbf{u} + \mathbf{a}, \mathbf{\Phi}] - \{\mathbf{u} + \mathbf{a}, \mathbf{u} + \mathbf{a}, \mathbf{\Phi}\} = 0 \tag{3.2}$$

in which the expression $[\mathbf{u}, \mathbf{v}]$ is defined by Eq. (2.2), and the following notation has been introduced:

$$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \int_\Omega \mathbf{u} \cdot \mathbf{v} \cdot \nabla \mathbf{w} dx$$

If a generalized solution $\mathbf{v} \in \mathbf{W}_2^2(\Omega')$ for any interior subdomain Ω' of Ω, then, there is a function $p(x)$, unique up to a constant term, such that $\nabla p \in \mathbf{L}_2(\Omega')$, and inequality (1.1) holds almost everywhere in Ω (an analogous assertion with regard to the generalized solution of the first boundary-value problem for (1.1) has been proved in [1]).

LEMMA 3.2. Let $\mathbf{v}(x)$ be a generalized solution of the auxiliary problem. Then, an upper bound $\|\mathbf{v}\|_2^{(1)}$ exists, depending only on w, δ, and l.

For the proof of the lemma, we put $\Phi = \mathbf{u}$ in the identity (3.2). Noting that for $\mathbf{u} \in H(\Omega)$, and the chosen $\mathbf{a}$

$$\{\mathbf{u}, \mathbf{a}, \mathbf{u}\} = 0, \quad \{\mathbf{u}, \mathbf{u}, \mathbf{u}\} = 0$$

we reduce relation (3.2) to the form

$$2\,|||\mathbf{u}|||^2 = \{\mathbf{a}, \mathbf{u}, \mathbf{u}\} + \{\mathbf{a}, \mathbf{a}, \mathbf{u}\} - 2\,[\mathbf{a}, \mathbf{u}] \tag{3.3}$$

Applying inequalities (3.1) to the estimate of the first term on the right-hand side and making use of inequality (2.5) and the Cauchy–Bunyakovskii inequality, we infer from (3.3) that

$$|||\mathbf{u}||| \leqslant 2\|\mathbf{a}\|_2^{(1)} + \sqrt{3}\,(\|\mathbf{a}\|_4)^2 = C_6 \tag{3.4}$$

Inasmuch as $\|\mathbf{v}\|_2^{(1)} \le \|\mathbf{u}\|_2^{(1)} + \|\mathbf{a}\|_2^{(1)}$, and the norms $\|\mathbf{u}\|_2^{(1)}$ and $|||\mathbf{u}|||$ are equivalent, Lemma 3.2 is thus proved.

THEOREM 3.1. At least one generalized solution of the auxiliary problem exists.

The theorem is proved by the method of Galerkin according to the scheme proposed by Fujita [10] in an investigation of the first boundary-value problem for the Navier–Stokes equations. For any $n \ge 1$, we construct an approximate solution of the problem in the form

$$\mathbf{v}_n = \mathbf{a} + \mathbf{u}_n \equiv \mathbf{a} + \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$$

The unknown coefficients α_i in the expansion of $\mathbf{u}_n$ on the basis $\{\mathbf{e}_i\}$ in $\mathbf{H}(\Omega)$ are determined from the conditions

$$2\,[\mathbf{u}_n + \mathbf{a}, \mathbf{e}_i] - \{\mathbf{u}_n + \mathbf{a}, \mathbf{u}_n + \mathbf{a}, \mathbf{e}_i\} = 0$$

for every $i = 1, \ldots, n$. As in [10], we establish an a priori estimate of $\mathbf{u}_n$ in $\mathbf{H}(\Omega)$, that is independent of n, and prove the existence of an approximate solution. The boundedness of $\mathbf{u}_n$ implies that there is a subsequence of $\mathbf{u}_n$ weakly convergent in $\mathbf{H}(\Omega)$. By the Sobolev embedding theorem (see, e.g., [4]), it converges strongly in $\mathbf{L}_4(\Omega)$. We readily infer from the latter fact that its weak limit $\mathbf{u} = \mathbf{v} - \mathbf{a}$ satisfies the identity (3.2), and thus, determines a generalized solution of the auxiliary problem.

THEOREM 3.2. If $|\mathbf{w}|_{2+\alpha[0,l]}$ is sufficiently small, the generalized solution of the auxiliary problem is unique.

The proof of the theorem is based on the derivation of the estimate $|||\mathbf{u}||| \le C_7\,|\mathbf{w}|_{2+\alpha}\,(1 + |w|_{2+\alpha})$, which follows from (3.4) and the properties of the continuation $\mathbf{a}$ of the vector $\mathbf{w}$. Otherwise, it follows the proof of the uniqueness theorem for slow steady-state flows (the first boundary-value problem) in [1].

We note in conclusion, that the existence and uniqueness theorems for the generalized solution of the auxiliary problem are also true in the case of weaker assumptions regarding the smoothness of $\mathbf{w}(x_1)$. Theorem 3.1 remains valid if $\mathbf{w} \in W_2^{1/2}(0, l)$, and in order for Theorem 3.2 to hold true, it is sufficient for $\|\mathbf{w}\|_2^{1/2}$ to be small.

4. Smoothness of the Generalized Solution

We now investigate the differential properties of the generalized solution of the auxiliary problem, and their dependence on the smoothness of the boundary Γ.

THEOREM 4.1. If $f \in C^{m+1+\alpha}\,[0, l]$, $\mathbf{w} \in \mathbf{C}^{m+\alpha}\,[0, l]$, $m \geqslant 2$, then, the generalized solution $\mathbf{v}$ of the auxiliary problem belongs to $\mathbf{C}^{m+\alpha}(\bar{\Omega})$, $\nabla p \in \mathbf{C}^{m-2+\alpha}(\bar{\Omega})$.

The proof is based on two lemmas concerning the solution of the linear problem (2.2) for the Stokes system (2.1).

LEMMA 4.1. If $\zeta \in L_r(\Omega)$, $r > 1$, and $f \in C^3[0, l]$, then, the corresponding solution **u** of problem (2.1)-(2.2) belongs to $W_r^{(2)}(\Omega)$, $\nabla p \in L_r(\Omega)$, and

$$\|u\|_r^{(2)} + \|\nabla p\|_r \leqslant C_8 \|\zeta\|_r \tag{4.1}$$

LEMMA 4.2. If $\zeta \in C^{m-2+\alpha}(\bar{\Omega})$, $f \in C^{m+1+\alpha}[0, l]$, $m \geqslant 2$, then, the solution **u** of problem (2.1)-(2.2) belongs to $C^{m+\alpha}(\bar{\Omega})$, $\nabla p \in C^{m-2+\alpha}(\bar{\Omega})$, and

$$|u|_{m+\alpha} + |\nabla p|_{m-2+\alpha} \leqslant C_9 |\zeta|_{m-2+\alpha} \tag{4.2}$$

We defer the proof of Lemmas 4.1 and 4.2 until the end of this section, showing for now how these lemmas imply the statement of Theorem 4.1.

Let **v** be the generalized solution of the auxiliary problem. Then, $u = v - a$ and the corresponding p satisfy the system (2.1) with $\zeta = \Delta a - v \cdot \nabla v$ and the homogeneous boundary conditions (2.2). According to Lemma 3.1, $\Delta a \in C^{m-2+\alpha}(\bar{\Omega})$. Due to Lemma 3.2, we have $v \in W_2^1(\Omega)$. By the embedding theorem [4], if Ω is a plane bounded domain with a piecewise-smooth boundary, then, the space $W_r^m(\Omega)$ is embedded in $L_q(\Omega)$ with $q = 2r/(2-rm)$ for $rm < 2$, and in $L_q(\Omega)$ with any finite q for $rm = 2$. Using this theorem with $r = 2$, $m = 1$, and the Hölder inequality, we infer that $v \cdot \nabla v \in L_r(\Omega)$ and, hence, $\zeta \in L_r(\Omega)$ with any finite r. From Lemma 4.1, we deduce the inclusion relation $v = u + a \in W_r^2(\Omega)$. Then, $\partial v/\partial x_i \in W_r^1(\Omega)$, $i = 1, 2$. Using the embedding theorem for $W_r^m(\Omega)$ in $C^h(\bar{\Omega})$, $0 < h < 1$, for $rm > n$, where n is the dimension of Ω [1], and choosing $r = 2/(1-\alpha)$, we find that $\partial v/\partial x_i \in C^{\alpha}(\bar{\Omega})$, so that $\zeta \in C^{\alpha}(\Omega)$. In accordance with Lemma 4.2, we infer that $v = u + a \in C^{2+\alpha}(\bar{\Omega})$. This proves Theorem 4.1, for the case $m = 2$. If $m > 2$, the same reasoning can be applied several times.

For the proof of Lemmas 4.1 and 4.2, we use a priori estimates for the solutions of system elliptic in the Douglis–Nirenberg sense [11]. An important algebraic condition has been formulated in [12, 13], namely a complementary condition guaranteeing the existence of ultimately sharp estimates in the norms of $C^{m+\alpha}$ and W_r^m for the indicated systems. We know from [13] that the Stokes system (2.1) is Douglis–Nirenberg elliptic. Straight-forward, though laborious, calculations show that the set of boundary conditions (2.2) for the system (2.1) satisfies the complementarity condition.

Let us suppose that under the conditions of Lemma 4.1 a solution $u \in W_r^{(2)}(\Omega)$ of problem (2.1)-(2.2) exists. Then, estimate (4.1) follows from the general results of Agmon, Douglis, and Nirenberg [12], and Solonnikov [14]. The absence from the right side of (4.1) of a term of the form $C\|u\|_r$ is attributable to the uniqueness theorem for problem (2.1)-(2.2): if $\zeta = 0$, then, $u = 0$, $p = \text{const}$. It suffices to verify the existence of a solution of (2.1)-(2.2) for $\zeta = C^{\infty}(\bar{\Omega})$, $f \in C^{\infty}[0, l]$. Due to estimate (4.1), it is possible by suitable approximations to then go over to the case $\zeta \in L_r$, $f \in C^3[0, l]$.

Introducing the stream function ψ by the relations $u_1 = \partial\psi/\partial x_2$, $u_2 = -\partial\psi/\partial x_1$, we can verify the fact that problem (2.1)-(2.2) is equivalent to the following:

$$\begin{gathered} \Delta\Delta\psi = \chi(x), \quad x \in \Omega \\ \psi(x_1 + l, x_2) \equiv \psi(x_1, x_2), \quad \psi|_{\Sigma} = 0, \ (\partial\psi/\partial n)|_{\Sigma} = 0 \\ \psi|_{\Gamma} = 0, \quad \Delta\psi - 2K(x)(\partial\psi/\partial n)|_{\Gamma} = 0 \end{gathered} \tag{4.3}$$

where $\chi = \partial\zeta_1/\partial x_2 - \partial\zeta_2/\partial x_1$, $\partial\psi/\partial n$ denotes the derivative in the direction of the outward normal to the boundary of Ω, and K is the curvature of Γ.

Problem (4.3) is self-adjoint. Its solution is unique, as can be proved by multiplication of the equation $\Delta\Delta\psi = 0$ by ψ, and integration by parts over the domain Ω with recognition of the boundary conditions. The system of boundary operators in (4.3) is normal and covers the operator $\Delta\Delta$ (for the definition of the latter, see [15]). The existence of a solution $\psi \in C^{\infty}$ of problem (4.3) is now a direct consequence of the results of Schechter [15]. This completes the proof of Lemma 4.1. The proof of Lemma 4.2 is analogous.

5. Supporting Lemmas

It follows from the results of Secs. 3 and 4 that if $f \in C^{3+\alpha}[0,l]$, $w \in C^{2+\alpha}[0,l]$, $|f|_{1+\alpha} \leqslant \delta < 1$ and $|w|_{2+\alpha} \leq \varepsilon$, where $\varepsilon > 0$ is small, then, the velocity **v** is uniquely determined in the solution of the auxiliary problem, and the corresponding pressure is determined by the curvilinear integral

$$p(x)=\int_0^x(\Delta\mathbf{v}-\mathbf{v}\cdot\nabla\mathbf{v})\cdot ds+p_0$$

in which $p_0 = p(0)$ is an arbitrary constant. This result permits us to bring in the operator $\mathcal{B}$, which associates with every l-periodic function $f \in C^{3+\alpha}$ a function $\mathcal{B}[f(x_1)]$ according to the relation

$$\mathcal{B}(f)=\mathbf{n}\cdot T|_{\Gamma}\cdot\mathbf{n}+p_0 \tag{5.1}$$

(here, $\mathbf{n}$ and $T|_{\Gamma}$ are treated as functions of x_1; we point out that the right-hand side of (5.1) does not depend on p_0). According to Theorem 4.1 the function $\mathcal{B}[f(x_1)]$, being a linear combination of p and $\partial v_i/\partial x_j$ with coefficients in $C^{2+\alpha}$, is a member of the class $C^{1+\alpha}[0, l]$. Moreover, this function is l-periodic in x_1. It is required to show that $\mathcal{B}(f)$ is continuous as an operator of $C^{3+\alpha}$ in $C^{1+\alpha}[0, l]$. We first state the following lemma.

LEMMA 5.1. Under the conditions of Theorem 4.1, the inequality

$$|\mathbf{v}|_{2+\alpha,\bar{\Omega}}+|\nabla p|_{\alpha,\bar{\Omega}}\leqslant C_{10}|\mathbf{w}|_{2+\alpha,[0,l]} \tag{5.2}$$

holds, where C_{10} does not depend on f or $\mathbf{w}$, if $|f|_{3+\alpha}\leq\delta$, $|\mathbf{w}|_{2+\alpha}\leq\varepsilon$.

We shall not give the proof of Lemma 5.1. It is based on the Schauder estimates for the solutions of elliptic systems [12, 14] and a repetition of the arguments in Sec. 4.

Next, we investigate the continuous dependence of the solution of the auxiliary problem on f, i.e., on the boundary Γ of the domain Ω. To do so, we transform to new independent variables ξ and η according to the relations

$$\xi=x_1,\quad \eta=\frac{x_2-f(x_1)}{1+f(x_1)}$$

The domain Ω in this case, is transformed into the rectangle $\Pi=\{\xi,\eta: 0<\xi<l, -1<\eta<0\}$. The system (1.1) is transformed to the following:

$$\begin{gathered}
\frac{\partial^2 u}{\partial\xi^2}-\frac{2(1+\eta)f'}{1+f}\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1+(1+\eta)^2f'^2}{(1+f)^2}\frac{\partial^2 u}{\partial\eta^2}+\\
+\left\{\frac{(1+\eta)[2f'^2-(1+f)f'']}{(1+f)^2}+\frac{(1+\eta)f'u}{1+f}-\frac{v}{1+f}\right\}\frac{\partial u}{\partial\eta}-\\
-u\frac{\partial u}{\partial\xi}-\frac{\partial q}{\partial\xi}+\frac{(1+\eta)f'}{1+f}\frac{\partial q}{\partial\eta}=0\\
\frac{\partial^2 v}{\partial\xi^2}-\frac{2(1+\eta)f'}{1+f}\frac{\partial^2 v}{\partial\xi\partial\eta}+\frac{1+(1+\eta)^2f'^2}{(1+f)^2}\frac{\partial^2 v}{\partial\eta^2}+\\
+\left\{\frac{(1+\eta)[2f'^2-(1+f)f'']}{(1+f)^2}+\frac{(1+\eta)f'u}{1+f}-\frac{v}{1+f}\right\}\frac{\partial v}{\partial\eta}-u\frac{\partial v}{\partial\xi}-\frac{1}{1+f}\frac{\partial q}{\partial\eta}=0\\
\frac{\partial u}{\partial\xi}-\frac{(1+\eta)f'}{1+f}\frac{\partial u}{\partial\eta}+\frac{1}{1+f}\frac{\partial v}{\partial\eta}=0
\end{gathered} \tag{5.3}$$

where

$$u(\xi,\eta)=v_1(x_1,x_2),\ v(\xi,\eta)=v_2(x_1,x_2),\ q(\xi,\eta)=p(x_1,x_2),$$
$$f(x_1)=f(\xi),\ f'=df/d\xi$$

The boundary conditions (1.2), (1.3), and (1.6) generate the following boundary conditions for the system (5.3):

$$\begin{gathered}
u(\xi+l,\eta)\equiv u(\xi,\eta),\quad v(\xi+l,\eta)\equiv v(\xi,\eta),\quad q(\xi+l,\eta)\equiv q(\xi,\eta)\\
u=w_1(\xi),\quad v=w_2(\xi)\ \text{for}\ \ \eta=-1\\
-2f'\frac{\partial u}{\partial\xi}+\frac{1+f'^2}{1+f}\frac{\partial u}{\partial\eta}+(1-f'^2)\frac{\partial v}{\partial\xi}+\frac{f'(1+f'^2)}{1+f}\frac{\partial v}{\partial\eta}=0,\\
f'u-v=0\ \text{for}\ \ \eta=0
\end{gathered} \tag{5.4}$$

The foregoing reduces the situation to an investigation of the continuous dependence of the solution of the boundary-value problem (5.3)-(5.4) in a fixed domain on the coefficients of the equations and the boundary conditions. We denote by $u^{(i)}$, $v^{(i)}$, $q^{(i)}$ ($i = 1, 2$), the solution of problem (5.3)-(5.4) with $f = f_i(\xi) \in C^{3+\alpha}$ · $[0, l]$.

LEMMA 5.2. If $|f_i|_{3+\alpha} \le \delta < 1$, $i = 1, 2$, and $|w|_{2+\alpha} \le \varepsilon$, where $\varepsilon > 0$ is small, the following estimate holds:

$$|u^{(1)} - u^{(2)}|_{2+\alpha,\bar{\pi}} + |v^{(1)} - v^{(2)}|_{2+\alpha,\bar{\pi}} + |\nabla (q^{(1)} - q^{(2)})|_{\alpha,\bar{\pi}} \leqslant C_{11} |w|_{2+\alpha,[0,l]} |f_1 - f_2|_{3+\alpha,[0,l]} \tag{5.5}$$

We merely outline the plan of the proof, inasmuch as the complete proof is rather bulky. We denote

$$u^* = u^{(1)} - u^{(2)}, \quad v^* = v^{(1)} - v^{(2)}, \quad q^* = q^{(1)} - q^{(2)}$$

Proceeding from (5.3) and (5.4), we obtain for u^*, v^*, and q^* a system of linear equations with linear boundary conditions. The right-hand sides of these equations and of the boundary conditions represent sums of products of the functions $u^{(1)}$, $v^{(1)}$, and $q^{(1)}$ or their derivatives by coefficients containing factors of the form $d^k(f_1 - f_2)/d\xi^k$, $k = 0, 1, 2$. For example, the last condition (5.4) yields the condition $f_2'u^* - v^* = -(f_1 - f_2)'u^{(1)}$.

The resulting linear system in u^*, v^*, and q^* is Douglis–Nirenberg elliptic, because its principal part is the Stokes system transformed to new independent variables, and this transformation preserves the ellipticity property [13]. The boundary-value problem for it satisfies the complementarity condition, because this condition is satisfied by the original problem. It is important to realize that for small ε, the new problem has a unique solution (this fact essentially follows from Theorem 3.2 on the uniqueness of the solution of the auxiliary problem for small $|\mathbf{w}|_{2+\alpha}$). The foregoing result enables us to deduce Schauder a priori estimates for u^*, v^*, and q^* directly in terms of the right-hand sides of the equations and the boundary conditions.

Using the results of [12, 14], we readily establish the following inequalities:

$$|u^*|_{2+\alpha,\bar{\Pi}} + |v^*|_{2+\alpha,\bar{\Pi}} + |\nabla q^*|_{\alpha,\bar{\Pi}} \leqslant C_{12} |f_1 - f_2|_{3+\alpha\ [0,l]}$$
$$\times (|u^{(1)}|_{2+\alpha,\bar{\Pi}} + |u^{(1)}|^2_{2+\alpha,\bar{\Pi}} + |v^{(1)}|_{2+\alpha,\bar{\Pi}} + |v^{(1)}|^2_{2+\alpha,\bar{\Pi}} + |\nabla q^{(1)}|_{\alpha,\bar{\Pi}})$$

Inasmuch as the mapping $(x_1, x_2) \to (\xi, \eta)$ belongs to class $C^{3+\alpha}$, we therefore, have $u^{(i)}, v^{(i)} \in C^{2+\alpha}(\bar{\Pi})$ $(i = 1, 2)$, $\nabla q \in C^{\alpha}(\bar{\Pi})$, and

$$|u^{(i)}|_{2+\alpha,\bar{\Pi}} + |v^{(i)}|_{2+\alpha,\bar{\Pi}} + |\nabla q^{(i)}|_{\alpha,\bar{\Pi}} \leqslant C_{13} (|\mathbf{v}^{(i)}|_{2+\alpha,\bar{\Omega}} + |\nabla p^{(i)}|_{\alpha,\bar{\Omega}})$$

The required estimate (5.5) follows from the last two inequalities and inequality (5.2).

LEMMA 5.3. Let the conditions of Lemma 5.2 be satisfied. Then, $\mathscr{B}(f) \in C^{1+\alpha}[0, l]$, and the following estimate holds:

$$|\mathscr{B}(f_1) - \mathscr{B}(f_2)|_{1+\alpha,[0,l]} \leqslant C_{14} |w|_{2+\alpha,[0,l]} |f_1 - f_2|_{3+\alpha,[0,l]} \tag{5.6}$$

The proof is based on the transformation to variables ξ, η, and the subsequent application of Lemma 5.2. Calculating the expression $\mathbf{n} \cdot T|_\Gamma \cdot \mathbf{n}$ in these variables, we obtain

$$\mathscr{B}[f(\xi)] = \Big[-(1 + f'^2) q + 2f'^2 \frac{\partial u}{\partial \xi} - \tag{5.7}$$

$$- \frac{2f'(1 + f'^2)}{1 + f} \frac{\partial u}{\partial \eta} - 2f' \frac{\partial v}{\partial \xi} + \frac{2(1 + f'^2)}{1 + f} \frac{\partial v}{\partial \eta}\Big] \quad \text{for} \quad \eta = 0$$

As mentioned, $u, v \in C^{2+\alpha}(\bar{\pi})$, $q \in C^{1+\alpha}(\bar{\pi})$. Therefore, $\mathscr{B}(f) \in C^{1+\alpha}[0, l]$. Determining the difference $\mathscr{B}(f_1) - \mathscr{B}(f_2)$ by (5.7) and invoking estimate (5.5), we arrive at the required inequality (5.6).

6. Determination of the Form of the Free Surface

The following theorem comprises the fundamental result of the article.

THEOREM 6.1. Let the function $\mathbf{w}(x_1)$ satisfy the conditions of Sec. 1, and let $|\mathbf{w}|_{2+\alpha}, [0, l] \le \varepsilon$, where $\varepsilon \ge 0$ is sufficiently small, then, a solution of the free-boundary problem (1.1)-(1.6) exists, such that $\mathbf{v} \in \mathbf{C}^{2+\alpha}(\bar{\Omega})$, $p \in C^{1+\alpha}(\bar{\Omega})$, $f \in C^{3+\alpha}[0, l]$. This solution is unique in a certain neighborhood of zero of the product space

$$C^{2+\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\bar{\Omega}) \times C^{3+\alpha}[0, l].$$

Proof. Under the conditions of the theorem there is a solution of the auxiliary problem, $\mathbf{v} \in C^{2+\alpha}(\bar{\Omega})$, $p \in C^{1+\alpha}(\bar{\Omega})$. In this case, $\mathbf{v}$ is uniquely determined for small ε, and p is determined up to a constant term p_0. From the solution of the auxiliary problem, we determine $\mathbf{n}\cdot T|_\Gamma\cdot\mathbf{n}$ as a function of $x_1 = \xi$, and substitute the result into condition (1.4). Using the notation (5.1), where $\mathcal{B}(f)$ is now uniquely determined, we write (1.4) in the form

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right)' = \mu\mathcal{B}(f) - \mu p_0 \tag{6.1}$$

The existence of an l-periodic solution $f(\xi)$ of Eq. (6.1) requires that the average value of the right-hand side of (6.1) on the interval $[0, l]$ be equal to zero. We therefore, find $p_0 = \overline{\mathcal{B}(f)}$, where $\overline{\mathcal{B}(f)}$ is the value of the function $\mathcal{B}[f(\xi)]$. Now, the pressure is automatically uniquely determined in the solution of the auxiliary problem. Theorem 6.1 will be proved, if we can find an l-periodic solution $f \in C^{3+\alpha}[0, l]$, with $\bar{f} = 0$ and show that this solution is unique, if $|f|_{3+\alpha}$ is sufficiently small.

Let $\varepsilon_1 > 0$ be so small that for $|\mathbf{w}|_{2+\alpha} \le \varepsilon_1$ inequality (5.6) is true and, in addition,

$$\mu \max_{\xi\in[0,l]} \left|\int_0^\xi \{\mathcal{B}[f(\tau)] - \overline{\mathcal{B}(f)}\}\,d\tau\right| \le \beta < 1$$

for any $f \in C^{3+\alpha}[0, l]$, such that $|f|_{3+\alpha} \le \delta$ and $\delta < 1$ is fixed. This choice of ε_1 is permissible (for fixed μ) by virtue of Lemma 5.1 and the definition of $\mathcal{B}(f)$. Then, Eq. (6.1) is transformed by twofold integration to the form $f = \mathcal{F}(f)$, where

$$\mathcal{F}(f) = \int_0^\xi \mu\int_0^\sigma \{\mathcal{B}[f(\tau)] - \overline{\mathcal{B}(f)}\}\,d\tau \times \left[1 - \left(\mu\int_0^\sigma\{\mathcal{B}[f(\tau)] - \overline{\mathcal{B}(f)}\}d\tau\right)^2\right]^{-1/2} d\sigma - f_0 \tag{6.2}$$

and f_0 is a constant equal to the average value on $[0, l]$ of the first term on the right-hand side of (6.2). We denote by N the subspace of $C^{3+\alpha}(-\infty, \infty)$ formed by l-periodic functions having zero-valued period averages. It follows from the definition of $\mathcal{F}$ and Lemma 5.3, that $\mathcal{F}(f) \in N$, if $f \in K_\delta$, where K_δ is the ball $|f|_{3+\alpha} \le \delta < 1$ in the space N, and

$$|\mathcal{F}(f)|_{3+\alpha} \le C_{15}|\mathbf{w}|_{2+\alpha}$$

where C_{15} is independent of $\mathbf{w}$, if $|\mathbf{w}|_{2+\alpha} \le \varepsilon_1$. Let $\varepsilon_2 = \min(\varepsilon_1, C_{15}^{-1}\delta)$, whereupon for $|\mathbf{w}|_{2+\alpha} \le \varepsilon_2$ the operation $\mathcal{F}(f)$ maps the ball K_δ into itself. From the definition (6.2) of the operator $\mathcal{F}$ and inequality (5.6), we deduce the estimate

$$|\mathcal{F}(f_1) - \mathcal{F}(f_2)|_{3+\alpha} \le C_{16}|\mathbf{w}|_{2+\alpha}|f_1 - f_2|_{3+\alpha} \tag{6.3}$$

for any $f_1, f_2 \in K_\delta$. We now put $\varepsilon = \min(\varepsilon_2, C_{16}^{-1}\beta)$, where $\beta > 0$ is any number less than unity. We infer on the basis of (6.3) that for $|\mathbf{w}|_{2+\alpha} \le \varepsilon$ the continuous operator $\mathcal{F}(f)$ is contractible in the ball K_δ, so that the equation $f = \mathcal{F}(f)$ has a unique solution in that ball. This proves the theorem.

7. Other Steady-State Free-Boundary Problems for the Navier–Stokes Equations

Other plane steady-state problems for the Navier–Stokes equations are investigated analogously on the assumption that the free boundary does not have points of contact with the bounding solid surfaces. We cite as an example the problem of steady-state periodic waves in a heavy liquid over a sloping periodic bottom. It is proved that if the bottom is sufficiently smooth and its angle of inclination with respect to the horizontal plane is sufficiently small, then, the solution is uniquely determined by specification of the mass flow or average depth of the liquid.

Another example is afforded by the problem of the steady motion of a fluid in the annular space between a rotating solid cylindrical surface and a free boundary on which the pressure is given as a function of the polar angle. It is proved that if this function is almost a constant, the solution of the problem is determined by specification of the area of the curvilinear annulus occupied by the fluid. In this case the motion is not necessarily slow; however, it must be close to the rotation of the fluid as an integral solid.

The technique described above for the analysis of plane problems involving a free boundary makes it possible to investigate certain three-dimensional problems as well. A representative example is the problem of a doubly periodic flow in a layer whose upper boundary is free and whose lower boundary is a solid plane investigated with a periodic alternation of ingress and egress zones. Here, the three-dimensional analogs of Theorems 3.1 and 3.2 on the existence and uniqueness of a generalized solution of the auxiliary problem are valid. If we postulate that this solution has Hölder-continuous second derivatives up to the free boundary, we can obtain results analogous to the lemmas and theorems of Secs. 4 and 5. In the final stage it is required, instead of Eq. (6.1), to solve an equation of the same type as the equation for minimal surfaces having a nonlocal operator on the right-hand side. Considering the velocity given on the bottom to be sufficiently small and adopting the above-indicated assumption with regard to the solution of the auxiliary problem, we can show that a doubly periodic free surface is uniquely determined in the small by specification of the average depth of the liquid.

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